<u>Affine connection</u> ∇ is a map:

$$\mathfrak{X}(M) \times \mathcal{T}(M) \ni (X,K) \longrightarrow \nabla_X K \in \mathcal{T}(M)$$

s.t.

1° $\nabla_X : \mathcal{T}(M) \to \mathcal{T}(M)$ preserves the type of tensor

2° $\nabla_{fX+gY} K = f\nabla_X K + g\nabla_Y K$

3° ∇ is $\mathbb{R}$-linear in K $\quad \nabla_X(\alpha K_1 + \beta K_2) = \alpha \nabla_X K_1 + \beta \nabla_X K_2$, $\alpha, \beta \in \mathbb{R}$

4° $\nabla_X (K \otimes L) = \nabla_X K \otimes L + K \otimes \nabla_X L$

5° ∇_X commutes with contractions

6° $\nabla_X f = X(f)$

$\nabla_X K$ - is called covariant derivative of K w.r.t. X

<u>Rmk 1</u> ∇ defines a map

$$\mathfrak{X}(M)^r_s \longrightarrow \mathfrak{X}(M)^r_{s+1}$$
$$K \longmapsto DK \quad \text{s.t.}$$

$$DK(X_1, \dots X_s, X) = (\nabla_X K)(X_1, \dots, X_s)$$

<u>Rmk 2</u> How to introduce connections on mfds?

Connection is a local notion in the following sense:

Let (X_μ) be a frame in U $\Rightarrow$

$$\boxed{\nabla_{X_\mu} X_\nu = \Gamma^\rho_{\nu\mu} X_\rho}$$

functions on U. (Smooth since X and ∇ is smooth)

$\Gamma^\rho_{\nu\mu}$ — these functions determine connection in U.

Let (ω^μ) be a coframe dual to (X_μ):

$$X_\mu \lrcorner\, \omega^\nu = \omega^\nu(X_\mu) = \delta^\nu_\mu$$

then:

contraction

$$[\nabla_{X_\mu}(\omega^\nu)](X_\varrho) = C^1_1(\nabla_{X_\mu}\omega^\nu \otimes X_\varrho) \overset{4^\circ}{=} C^1_1\, \nabla_{X_\mu}(\omega^\nu \otimes X_\varrho) - C^1_1(\omega^\nu \otimes \nabla_{X_\mu} X_\varrho)$$

$$\overset{5^\circ}{=} \nabla_{X_\mu} C^1_1(\omega^\nu \otimes X_\varrho) - C^1_1(\omega^\nu \otimes \Gamma^\alpha{}_{\varrho\mu} X_\alpha) = \cancel{\nabla_{X_\mu}\delta^\nu_\varrho} - \Gamma^\alpha{}_{\varrho\mu}\, C^1_1(\omega^\nu \otimes X_\alpha) =$$

$$= -\Gamma^\nu{}_{\varrho\mu}$$

$$\Rightarrow \boxed{\nabla_{X_\mu}\omega^\nu = -\Gamma^\nu{}_{\varrho\mu}\,\omega^\varrho}$$

Now: useful notation: (X_μ) - a frame in U (not necessarily holonomic!)

$$\nabla_{X_\mu} K := \nabla_\mu K$$

Then, for a general tensor field

$$K = K^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}\, X_{\mu_1}\otimes\ldots\otimes X_{\mu_r}\otimes\omega^{\nu_1}\otimes\ldots\otimes\omega^{\nu_s}$$

if Z is a vector field $Z = Z^\mu X_\mu$ then

$$\boxed{\nabla_Z K \overset{2^\circ}{=} Z_\varrho \nabla_\varrho K = Z_\varrho \underline{(\nabla_\varrho K)^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}}\, X_{\mu_1}\otimes\ldots\otimes X_{\mu_r}\otimes\omega^{\nu_1}\otimes\ldots\otimes\omega^{\nu_s}}$$

Where:

(CD)
$$\boxed{\begin{aligned}(\nabla_\varrho K)^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} &= X_\varrho(K^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}) + \\ &+ \sum_{i=1}^{N} \Gamma^{\mu_i}{}_{\sigma\varrho} K^{\mu_1\ldots\underset{\substack{\uparrow\\ \text{ith place}}}{\sigma}\ldots\mu_r}{}_{\nu_1\ldots\nu_s} - \sum_{i=1}^{s}\Gamma^\sigma{}_{\nu_i\varrho} K^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\overset{\substack{\text{ith place}\\ \downarrow}}{\sigma}\ldots\nu_s}\end{aligned}}$$

Formula (CD), once $\Gamma^{\mu}{}_{\nu\rho}$ are specified, determines ∇ in U.

- change of basis

$$X'_\mu = X_\nu a^{-1\nu}{}_\mu \iff X_\mu = a^\nu{}_\mu X'_\nu$$

thus

$$\Gamma^\rho{}_{\nu\mu} X_\rho = \nabla_{X_\mu} X_\nu = \nabla_{a^\alpha{}_\mu X'_\alpha}\left(a^\beta{}_\nu X'_\beta\right) =$$

$$= a^\alpha{}_\mu \nabla_{X'_\alpha}\left(a^\beta{}_\nu X'_\beta\right) = a^\alpha{}_\mu \left[X'_\alpha(a^\beta{}_\nu) X'_\beta + a^\beta{}_\nu \Gamma'^\sigma{}_{\beta\alpha} X'_\sigma\right]$$

$$= X_\mu(a^\beta{}_\nu) X'_\beta + a^\alpha{}_\mu a^\beta{}_\nu \Gamma'^\sigma{}_{\beta\alpha} X_\rho a^{-1\rho}{}_\sigma =$$

$$= X_\mu \lrcorner\, da^\beta{}_\nu\, a^{-1\rho}{}_\beta X_\rho + a^\alpha{}_\mu a^\beta{}_\nu \Gamma'^\sigma{}_{\beta\alpha} a^{-1\rho}{}_\sigma X_\rho$$

$$\boxed{\Gamma^\rho{}_{\nu\mu} = a^\alpha{}_\mu a^\beta{}_\nu \Gamma'^\sigma{}_{\beta\alpha} a^{-1\rho}{}_\sigma + a^{-1\rho}{}_\beta X_\mu(a^\beta{}_\nu)}$$

Trick: Connection 1-forms in frame (X_μ)

$$\Gamma^\rho{}_{\mu\nu} \rightsquigarrow \Gamma^\rho{}_{\mu\nu}\,\omega^\nu =: \Gamma^\rho{}_\mu$$

$$\boxed{\Gamma^\rho{}_\nu = a^\beta{}_\nu \Gamma'^\sigma{}_\beta a^{-1\rho}{}_\sigma + a^{-1\rho}{}_\beta\, d(a^\beta{}_\nu)}$$

in matrix notation

$$\Gamma = a^{-1}\Gamma' a + a^{-1} da$$

$$\Rightarrow \quad \Gamma' = a\Gamma a^{-1} - da \cdot a^{-1}$$

but $\underbrace{d(a a^{-1})}_{\overset{\shortparallel}{0}} = da\, a^{-1} + a\, da^{-1}$

$$\boxed{\Gamma' = a\Gamma a^{-1} + a\, da^{-1}} \quad \text{or}$$

$$\boxed{\Gamma'^{\mu}{}_{\nu} = a^{\mu}{}_{\alpha}\Gamma^{\alpha}{}_{\beta}a^{-1\beta}{}_{\nu} + a^{\mu}{}_{\alpha}\, da^{-1\alpha}{}_{\nu}} \quad (TC)$$

Connections transform differently then tensors!
These are different kind of objects!

Now having two frames X_μ and X'_μ on two open sets $\mathcal{U}$ and $\mathcal{U}'$ with $\mathcal{U}\cap\mathcal{U}' \neq \emptyset$ we can take $\Gamma^{\mu}{}_{\nu}$ in $\mathcal{U}$ and $\Gamma'^{\mu}{}_{\nu}$ in $\mathcal{U}'$. They define <u>the same</u> connection in $\mathcal{U}\cup\mathcal{U}'$ provided the there exists $GL(n,\mathbb{R})$-valued function a on $\mathcal{U}\cap\mathcal{U}'$ so that Γ' and Γ are related by (TC).

<u>Rmk 3</u> How the notion of connection was obstracted?

Digression à la Cartan

$G = GL(n,\mathbb{R})$ acts on the space of all local frames in $\mathcal{U}$

$$(X_\mu) \xmapsto{a} (X'_\mu) = (X_\nu\, a^{-1\,\nu}{}_\mu)$$

It also acts on the space of all coframes in $\mathcal{U}$

$$(\omega^\mu) \xmapsto{a} (\omega'^\mu) = (a^\mu{}_\nu\, \omega^\nu)$$

$$\omega \longmapsto \omega' = a\omega .$$

If we have $W = \mathbb{R}^N$ and representation

$$\rho : GL(n,\mathbb{R}) \to GL(N,\mathbb{R})$$

we define:

<u>k-form of type ρ in $\mathcal{U}$</u> is an assignment:

$$\alpha : \omega \longmapsto \alpha(\omega) \in W \otimes \Lambda^k \mathcal{U}$$

s.t. $\quad \alpha(a\omega) = \rho(a)\alpha(\omega).$

<u>Example</u>

1) $W = \mathbb{R}^n$, $\rho = \mathrm{id}$, i.e. $\rho(a) = a$, $k = 1$

'moving coframe'

$\theta = (\theta^\mu) \quad \mu = 1, \dots, n \qquad\qquad \theta$ - 1-form of type id.

$$\theta^\mu(\omega) := \omega^\mu$$

$$\theta^\mu(a\omega) = a^\mu{}_\nu\, \omega^\nu .$$

2) $W = \mathbb{R}^n$, $\rho = id$, $k = 0$

X - vector field.

$$X^\mu(\omega) = X \lrcorner \omega^\mu$$

$$X^\mu(a\omega) = X \lrcorner (a^\mu{}_\nu \omega^\nu) = a^\mu{}_\nu (X \lrcorner \omega^\nu) = a^\mu{}_\nu X^\nu(\omega)$$

$\Rightarrow$ (components of vector-fields) $\rightsquigarrow$ (0-forms of type id)

3) tensors $\rightsquigarrow$ 0-forms of type $\rho^r{}_s$.

4) scalar forms $\equiv$ forms

$W = \mathbb{R}^1$, $\rho(a) = 1$.

Differentiation of forms of type ρ.

$X^\mu(\omega)$. $\rightsquigarrow$ $dX^\mu(\omega)$

what object is this?

$$dX^\mu(a\omega) = d\left(a^\mu{}_\nu X^\nu(\omega)\right) =$$

$$= a^\mu{}_\nu \, d(X^\nu(\omega)) + \underbrace{da^\mu{}_\nu \, X^\nu(\omega)}$$

this term makes $dX^\mu(a\omega)$ an object beyond the class of forms of types ρ.

In order to define differentiation that transforms objects of type ρ into objects of type ρ one introduces

$$\Gamma^\mu{}_\nu(\omega).$$

We want that

$$dX^\mu(a\omega) + \underbrace{\Gamma^\mu{}_\nu(a\omega)}_{\substack{\text{matrix-valued} \\ 1\text{-form}}} X^\nu(a\omega) = a^\mu{}_\nu\left(dX^\nu(\omega) + \Gamma^\nu{}_\rho(\omega) X^\rho(\omega)\right)$$

$$\parallel$$

$$da^\mu{}_\nu X^\nu(\omega) + a^\mu{}_\nu dX^\nu(\omega) + \Gamma^\mu{}_\nu(a\omega) a^\nu{}_\rho X^\rho(\omega)$$

$$\Gamma(a\omega)a + da = a\Gamma(\omega)$$

$$\boxed{\Gamma(a\omega) = a\Gamma(\omega)a^{-1} - da\,a^{-1}} \quad (\mathrm{TR})$$

Affine connection on M is an assigment

$$\Gamma : \omega \longrightarrow \Gamma(\omega) \in \mathrm{End}(\mathbb{R}^n) \otimes \Lambda^1 M$$

in such a way that if $\omega \to a\omega$ then (TR) for Γ.

$$(DX^\mu)(\omega) = dX^\mu(\omega) + \Gamma^\mu{}_\nu(\omega) X^\nu(\omega)$$

$\boxed{DX^\mu = dX^\mu + \Gamma^\mu{}_\nu X^\nu}$ ← covariant differential.